THE WINTER MEETING OF THE CHICAGO SECTION.

A REGULAR meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held in the Central High School Building at St. Louis, in connection with the American Association for the Advancement of Science, on Thursday and Friday, December 31, 1903, and January 1, 1904. The following members were present:

Professor C. H. Ashton, Professor H. T. Eddy, Professor G. W. Greenwood, Professor E. R. Hedrick, Professor Thomas F. Holgate, Professor J. A. Miller, Dr. H. L. Rietz, Professor D. A. Rothrock, Mr. Oscar Schmiedel, Professor J. B. Shaw, Professor Ormond Stone, Mr. E. H. Taylor, Professor C. A. Waldo, Professor L. G. Weld, Professor H. S. White, Professor J. N. Van der Vries.

At the first session Professor Ormond Stone presided, Professor Weld occupying the chair during the remainder of the meeting. On the afternoon of the first day a visit was paid to Washington University and the grounds of the St. Louis Exposition, through the kindness of the Commissioners of the Exposition.

The following officers were elected for the ensuing year: Secretary, Professor Thomas F. Holgate; Members of the programme committee, Dr. S. E. Slocum and Dr. G. A. Bliss.

The report of the committee appointed at the meeting of the section held in January, 1902, to consider the requirements for the master's degree for candidates making mathematics their major subject, was taken up for discussion and after some amendments was recommended for publication. The report will appear in the BULLETIN.

The following papers were read:

(1) Professor E. R. HEDRICK: "The law of the mean for functions of several variables."

(2) Professor E. W. DAVIS: "The elliptic functions and the general symmetric group on four letters."

(3) Professor G. A. MILLER: "On the generalization and extension of Sylow's theorem."

(4) Mr. OSCAR SCHMIEDEL: "Analogues of the jacobian identity that involve four elements."

(5) Professor H. S. WHITE: "Linear systems of hyperelliptic plane curves of the first kind."

(6) Professor E. R. HEDRICK: "A class of pseudo-contact transformations."

(7) Mr. J. V. COLLINS: "Some developments in vector analysis."

(8) Professor J. B. SHAW: "Algebras defined by finite groups."

(9) Dr. H. L. RIETZ: "Groups in which certain commutative operations are conjugate; and complete sets of conjugate operations."

(10) Professor L. E. DICKSON: "A generalization of symmetric and skew-symmetric determinants."

(11) Professor Jacob WESTLUND: "Primitive roots of an ideal in an algebraic number field."

(12) Dr. G. A. BLISS: "An existence theorem for a differential equation of the second order, with an application to the calculus of variations."

(13) Dr. SAUL EPSTEEN: "The definition of a reducible hypercomplex number system."

(14) PROFESSOR L. E. DICKSON: "Memoirs on abelian transformations."

(15) Mr. H. E. JORDAN: "Group characters of the linear fractional and binary linear homogeneous groups of determinant unity in any Galois field."

(16) Mr. H. E. JORDAN: "Group characters of the group of all linear fractional substitutions in any Galois field."

(17) Mr. J. J. QUINN: "A linkage for describing the conic sections by continuous motion."

(18) Mr. T. R. RUNNING: "Circles represented by the equation $u^3P + Lu^2Q + MuR + NS = 0$."

The papers of Professor Miller, Mr. Quinn, and Mr. Running were carried over from the programme of Section A of the American Association. The first was presented by Professor Weld, the last two were read by title. Abstracts of these papers were included in Professor Weld's report of the meeting of Section A, in the March BULLETIN, pages 291–293. Professor Davis's paper was presented by Professor White, and that of Mr. Collins by Professor Shaw. In the absence of the authors the papers by Professor Dickson, Professor Westlund, Dr. Bliss, Dr. Epsteen and Mr. Jordan were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The law of the mean for functions of one variable is known to hold, even when the derivative of the function is discontinuous. For functions of two variables, the law of the mean is usually stated for functions which have continuous derivatives, and the common form when the derivatives are discontinuous involves two undetermined numbers θ, θ'. In Professor Hedrick's paper a form is derived which holds when the function has discontinuous derivatives, and which involves only one undetermined number θ. Taking

$$\phi(t) = f(x + ht, y + k) + f(x, y + kt),$$

where $f(x, y)$ is the given function of the two variables x and y, it is easy to show that the derivative of $\phi(t)$ exists, provided the derivatives of $f(x, y)$ merely exist. It follows that the law of the mean for a single variable applies, and hence

$$\phi(1) - \phi(0) = \phi'(\theta), \qquad (0 < \theta < 1).$$

The substitution of the value of $\phi(t)$ in this form gives the desired formula

$$f(x + h, y + k) - f(x, y) = hf_x(x + \theta h, y + k) + kf_y(x, y + \theta k).$$

This formula still involves two distinct points, and these points lie on the sides of the rectangle whose corners are (x, y) and $(x + h, y + k)$. It would seem desirable to find a formula which should involve only one point, and that a point inside the rectangle.

2. Professor Davis's paper starts from the standard formula

$$\int_{\alpha}^{x} \frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)}} = \frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}} \operatorname{sn}^{-1} \sqrt{\frac{(\beta-\delta)(x-\alpha)}{(\alpha-\delta)(x-\beta)}}, \qquad \text{mod } \sqrt{\frac{(\beta-\gamma)(\alpha-\delta)}{(\alpha-\gamma)(\beta-\delta)}}$$

and classifies the various changes produced by the permutations of α, β, γ and δ. This is made easy and interesting by the use of Glaisher's notation *sc*, *cs*, *sd*, *ds*, etc.

4. Mr. Schmiedel presented some identities involving four elements, analogous to the jacobian identity with three. If Uf is the symbol of infinitesimal transformation, the jacobian identity is written

$$[(U_1U_2)U_3] + [(U_2U_3)U_1] + [(U_3U_1)U_2] = 0;$$

or, retaining indexes only, $[(12)3] + [(23)1] + [(31)2] = 0$; or, in abbreviated form, $(1, 2, 3) = 0$. Now the analogues with four elements consisting of terms derived from each other by cyclic permutation are $(1, 2, 3, 4) = [(13)(24)]$, together with twenty-three others obtained from this by interchange of figures. If the sum in the first member is to consist of terms derivable from each other by advancing each element by unity, another set of twenty-four identities is obtained of which the first member is this sum, and the second $\pm[(13)(24)]$, or twice this, according to the permutation.

5. Of hyperelliptic curves in a plane there are two principal classes: (1) Those whose passage through one point does not involve passage through a definite corresponding point and (2) systems in which there does occur such a pairing of points. In a linear system of the first kind the adjoints are degenerate and are equivalent to sets of right lines; hence the system is reducible by Cremona transformations to the curves of order m having a common multiple point of order $m - 2$. (This theorem is found to be due to Castelnuovo, 1890.) Systems of the second kind are "involution curves" of an involution of order 2, and can be generated from rational systems by transformations (due to Clebsch and Noether) of the three types which relate a double plane of deficiency zero to a simple plane. In his paper Professor White shows typical reduced systems for odd deficiency.

6. An ordinary contact transformation may be defined as a transformation of all the direction elements in a plane (or in space), which carries any curve (or surface in space) into a curve (or surface). Restricting ourselves, for simplicity, to the plane, it is evident that when we are dealing with a specialized family of curves only, more general transformations must exist, which carry the curves of that peculiar family into curves. These transformations will be called *pseudo-contact transformations*. In Professor Hedrick's second paper it is shown that,

given *any* transformation $X = f(x, y, y')$, $Y = \phi(x, y, y')$; of the direction elements of one plane into the points of another, a pseudo-contact transformation is determined, which carries the solutions of a certain differential equation into curves. This transformation is useful in the study of this particular family of curves, and in particular, in the study of this differential equation. Conversely, given any differential equation of the first order, or any one-parameter family of curves, there exist an infinite number of such pseudo-contact transformations which are useful in the study of the differential equation, or of the family. Examples of such transformations are given in the paper, and the extension to space is pointed out.

7. Mr. Collins showed how not only quaternions, but also other similar branches of vector analysis can be developed quite simply from certain fundamental laws.

8. Professor Shaw's paper develops the general method of reducing any abstract group, when considered as a linear associative algebra, to its expression in canonical form.* The general form of any expression in an algebra of this kind is $\Sigma a_{r,0}\, \lambda_{r,0}$. The units $\lambda_{r,0}$ fall into p sets, the tth set containing μ_t^2 forms, so that, n being the order of the group,

$$n = \mu_1^2 + \mu_2^2 + \cdots + \mu_p^2.$$

The general theory is then applied to several examples. The forms of the canonical units are developed for abelian groups, dihedral groups, the symmetric group on three letters, the tetrahedral group, and the octahedral group. In conclusion, the very intimate connection between this theory and Frobenius' theory of group characters and group determinants, and Dickson's theory of groups defined for an abstract field, is discussed. Also the place of this development in the general theory of linear associative algebra is fixed.

9. In Dr. Rietz's paper it is first shown that the group of order two and the symmetric group of order six are the only groups in which every two commutative operations, neither of which is identity, belong to the same conjugate set of operations. The nature of the groups is next examined, which have the property

* *Transactions*, vol. 4 (1903), pp. 251–287.

that any two commutative operations of the same order belong to the same conjugate set. It is shown, among other things, that these groups are either perfect groups or are isomorphic to the group of order 2^{α} all of whose operations except identity are of order 2, and that in the latter case the commutator subgroup corresponds to identity in the quotient group of order 2^{α}. If, in the group under consideration, all operations of the same order are conjugate, α is not greater than unity.

A closely related problem is then considered. Inquiry is made as to what groups exist in which the number λ of complete sets of conjugate operations, and the number n of distinct primes in the order are connected by a certain simple relation. No groups are possible with $\lambda < n + 1$. The group of order 2 and the symmetric group of order 6 are the only groups with $\lambda = n + 1$. The group of order 3, the non-cyclic group of order 10, the alternating group of order 12, and the simple group of order 60 are the only groups with $\lambda = n + 2$.

10. Professor Dickson's first paper considers the determinant $A = |\alpha_{ij}|$ where a_{ji} and a_{ij} are conjugate imaginaries for every $i, j = 1, \cdots, m$. It follows readily that A has a real value. Set $a_{rr} = a_{rr} + iA_{rr}$. For $m = 3$, $A - |a_{rs}|$ is a ternary quadratic form in A_{12}, A_{13}, A_{23}, whose determinant equals $|a_{rs}|$. For $m = 4$, $A - |a_{rs}| - S^2$ is a quadratic form in the six quantities A_{rs} whose determinant is the second compound of $|a_{rs}|$, where S is the Pfaffian $A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23}$. An interpretation in line coördinates is thus evident.

The chief claim for attention to the symmetrically conjugate determinant A is the elegance with which it expresses one of the concomitants (Begleitform) of the quadratic form $\Sigma a_{rs}x_rx_s$.

The paper appeared in the December number of the *American Mathematical Monthly*.

11. If A be a prime ideal in an arbitrary algebraic number field, there always exist primitive roots of A. In Professor Westlund's paper the question whether there exist primitive roots of any ideal is discussed, and the necessary and sufficient conditions for the existence of such primitive roots are determined

12. Picard has shown that two points p and q can be joined by a uniquely determined solution of the equation

$$\frac{d^2y}{dx^2} = f\left(x,\ y,\ \frac{dy}{dx}\right),$$

provided that q lies in a suitably chosen region adjoining p. Dr. Bliss discusses the nature of this solution considered as a function of the coördinates of p and q, and applies the results to prove an existence theorem for the equation

$$\frac{1}{\rho} = G\left(x,\ y,\ \frac{dx}{dt},\ \frac{dy}{dt}\right),$$

where G is a positively homogeneous function of dx/dt and dy/dt, and ρ is the radius of curvature of the solution $x = x(t)$, $y = y(t)$. This theorem is in turn applied to show that if the two points p and q are near enough to each other, they can always be joined by a uniquely determined extremal of the calculus of variations problem, corresponding to the integral

$$I = \int_{t_0}^{t_1} F\left(x,\ y,\ \frac{dx}{dt},\ \frac{dy}{dt}\right) dt,$$

and this extremal minimizes I between p and q.

13. According to Benjamin Peirce (*American Journal*, volume 4, page 100) and Scheffers (*Mathematische Annalen*, volume 39, page 317), a hypercomplex number system

$$E \equiv E_j E_k \equiv l_1 \cdots l_m l_{m+1} \cdots l_n$$

is reducible when the following conditions are fulfilled: C_1, E_j forms a system by itself; C_2, E_k forms a system by itself; A, $ll_k = 0$; B, $l_k l_j = 0$ $(j = l, \cdots m;\quad k = m+1, \cdots n)$.

The chief result of Dr. Epsteen's paper is to show that for systems containing a modulus the conditions C_1, C_2 are consequences of A, B; hence, a hypercomplex number system, containing a modulus is reducible if $ll_k = l_k l_l = 0$.

The paper will be submitted for publication in the *Transactions*.

14. A complete theory of canonical forms of linear transformations in an arbitrary field has been developed and applied, for example, to the distribution into sets of conjugates of the transformations of the general linear group. When we pass

from the general to a particular linear group, we encounter an incomparably more difficult problem. For the quaternary abelian group in a Galois field, an investigation was given in the *Transactions*, volume 2 (1901), pages 103–138. Professor Dickson's second paper aims at a systematic method for the $2m$-ary abelian group in a general field. The results are complete for $m = 1, 2, 3$, and are applied to the distribution of the operators into sets of conjugates.

In view of the holoedric isomorphism of the senary abelian group G, modulo 2, with the group of the 28 bitangents to a quartic without double points, the author gives (as an illustration of the results of the memoir) the following table showing the period of any transformation T of a chosen complete set of conjugates and the number of transformations of G commutative with T:

Period.	Commutative.	Period.	Commutative.	Period.	Commutative.
1	all	4	$2^6.3$	6	$2^4.3^2$
2	$2^7.3$	4	2^7	7	7
2	$2^6.3$	4	$2^7.3$	8	2^4
2	$2^5.3^2$	5	2.3.5	8	2^5
2	$2^6.3^2.5$	6	$2^4.3$	9	3^2
3	$2^2.3^3$	6	$2^4.3^2$	10	2.5
3	$2^3.3^4$	6	$2^2.3^2$	12	$2^2.3$
3	$2^4.3^3.5$	6	$2^3.3^2$	12	$2^3.3$
4	2^5	6	$2^4.3$	12	$2^2.3$
4	$2^6.3$	6	$2^4.3^2$	15	3.5

The paper will appear in the *American Journal of Mathematics*.

15, 16. In Mr. Jordan's papers the group characters for each of the three systems of groups are determined. They are about as simple as those obtained by Frobenius (*Berliner Sitzungsberichte*, 1896, pages 985–1021) for the linear fractional group of determinant unity modulo p. The case $p^n = 2^n$ requires separate treatment. Interesting relations hold between the characters of a group and those of one of its quotient groups.

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ON DOUBLY INFINITE SYSTEMS OF DIRECTLY SIMILAR CONVEX ARCHES WITH COMMON BASE LINE.

BY PROFESSOR E. H. MOORE.

In his paper "The determination of the constants in the problem of the brachistochrone" (BULLETIN, January, 1904, pages 185–188) Professor Bolza has proved analytically the statement of Weierstrass (lectures of 1882) that, of all cycloid arches having bases in a given line and lying on one side of that line, precisely one passes through any two points, lying on that side of the line or one or both lying on the line, and not lying in the same perpendicular to the line.

The special case in which one of the points is on the base line was handled geometrically by the brothers Bernoulli (cf. Ostwald's "Klassiker der exacten Wissenschaften," number 46, pages 12 and 18), use being made of the fact that all cycloid arches are similar. In reading his paper before the mathematical club of the University of Chicago, November 20, 1903, Professor Bolza, after indicating this solution, added a geometric solution for the case in which the segment is parallel to the base line, use being made furthermore of the fact that a cycloid arch is symmetric with respect to the perpendicular bisector of its base.

In the present note I give an analytically phrased geometric proof of the statement of Weierstrass in its generality, at the same time extending it (cf. theorem I_0, §3) to cover the case of *any doubly infinite system of directly* similar convex arches possessing tangents and meeting perpendicularly a common base line.*

One obtains an obvious generalization of this theorem in replacing perpendicularity to the base line by parallelism to any given line not parallel to the base line. More essential generalizations remain for consideration. In a subsequent note I propose to take up the connections of this theorem with the calculus of variations.

* The distinction between direct and inverse similarity enters only in case the arches do not have the symmetry spoken of above.

§ 1. *The αβ-Arches.*

With respect to a rectangular xy coördinate system, a curve-arc AB joining two points $A=(a,\ 0)$, $B=(b,\ 0)$, where $a<b$, is called an $\alpha\beta$-arch, in case its equation

$$y=f(x)$$

satisfies the following conditions 1°–4°.

1°. $\qquad a\leqq x\leqq b.$

2°. For every x, $a\leqq x\leqq b$, $f(x)$ is a single valued continuous function of x with a definite (finite or infinite) derivative * $f'(x)$.

3°. If $f'(x_1)=f'(x_2)$, $\ a\leqq x_1\leqq b$, $\ a\leqq x_2\leqq b$, then $x_1=x_2$.

4°. $\qquad f'(a)=\alpha\geqq 0,\quad f'(b)=\beta.$

The theorem of the mean is applicable to every arc of an $\alpha\beta$-arch. An $\alpha\beta$-arch AB is readily proved to have the following properties 5°–11°.

5°. A straight line having two distinct points in common with the arch is tangent to the arch at neither of the points and has no third point in common with the arch.

6°. $\qquad +\infty\geqq\alpha>0>\beta\geqq-\infty.$

7°. $\qquad f(x)>0 \quad \text{for} \quad a<x<b.$

8°. The transformation

$$R:\qquad \bar{x}=-x,\quad \bar{y}=y,$$

changes the $\alpha\beta$-arch AB into an $\bar{\alpha}\bar{\beta}$-arch $\bar{B}\bar{A}$, where

$$\bar{\alpha}=-\beta,\quad \bar{\beta}=-\alpha.$$

9°. The transformation

$$T_\kappa:\qquad \bar{x}=x,\quad \bar{y}=y-\kappa(x-a)\qquad (\alpha>\kappa\geqq 0),$$

changes the arc AC of the $\alpha\beta$-arch AB into an $\bar{\alpha}\bar{\beta}$-arch $\bar{A}\bar{C}$, with

$$\bar{\alpha}=\alpha-\kappa,\quad \bar{\beta}=\gamma-\kappa,$$

* The understanding is that the derivatives $f'(a)$, $f'(b)$ are respectively progressive, regressive, and that the definite infinite derivatives $+\infty$, $-\infty$ count as distinct.

where C is the definite point $(c, f(c))$, $a < c \leqq b$, common to the arch AB and the line $y = \kappa(x - a)$, and where $\gamma = f'(c)$.

10°. The arch AB has a single maximum point $M = (m, f(m))$, $f(m) > 0, f'(m) = 0$, $a < m < b$. A line $y = \eta$, $0 \leqq \eta \leqq f(m)$, meets the arch precisely once on each arc AM, MB, say at $P_{0\eta} = (x_{0\eta}, \eta)$, $P_{1\eta} = (x_{1\eta}, \eta)$, so that $a \leqq x_{0\eta} \leqq m \leqq x_{1\eta} \leqq b$. It is tangent to the arch only in case $\eta = f(m)$. On the arc AM, y is a continuous monotonic increasing function of x, since otherwise the arch would have two tangents parallel to the x-axis, and hence x is such a function of y. On the arc MB, x and y are continuous monotonic decreasing functions each of the other. Hence on the interval $0 \leqq \eta \leqq f(m)$, the continuous monotonic function $x_{1\eta} - x_{0\eta}$ decreases from $b - a$ to 0, while the quotient $\eta/(x_{1\eta} - x_{0\eta})$, continuous except for $\eta = f(m)$ and monotonic, increases from 0 to $+\infty$, and hence assumes precisely once any particular finite value $\lambda \geqq 0$.

11°. On the interval $a \leqq x \leqq b$ $f'(x)$ is a continuous monotonic function decreasing from α to β.—This follows readily from 8°, 9°, 10°.

§ 2. *The Group G_2 and the $\kappa\lambda$-Segments.*

The transformations

$$T_{k,l}: \qquad \bar{x} = kx + l, \quad \bar{y} = ky \qquad (k > 0)$$

constitute a two parameter group, G_2, generated by the translations $T_{1,l}$ and the expansions $T_{k,0}$.

Under a transformation $T_{k,l}$ an $\alpha\beta$-arch changes into an $\alpha\beta$-arch, identical with the former only if $T_{k,l}$ is the identity $T_{1,0}$. Hence under the group G_2 an $\alpha\beta$-arch changes into ∞^2 $\alpha\beta$-arches constituting a two-parameter system of such arches equivalent under the group.

A pair of points

$$P_0 = (x_0, y_0), \quad P_1 = (x_1, y_1) \qquad (x_0 < x_1, y_0 \geqq 0, y_1 \geqq 0)$$

determines a so-called $\kappa\lambda$-segment P_0P_1, where

$$y_1 - y_0 = \kappa(x_1 - x_0), \quad y_1 + y_0 = 2\lambda(x_1 - x_0).$$

The constants κ, λ are characteristic invariants of the segment P_0P_1 under the group G_2, that is, if P_0P_1 and $\bar{P}_0\bar{P}_1$ are equivalent under G_2, then $\kappa = \bar{\kappa}$, $\lambda = \bar{\lambda}$, and, conversely, if

$\kappa = \bar{\kappa}$, $\lambda = \bar{\lambda}$ then P_0P_1 and $\bar{P}_0\bar{P}_1$ are equivalent under G_2, indeed by means of a single transformation $T_{h,l}$ of G_2.

These characteristic invariants κ, λ are subject merely to the conditions $\kappa =$ any, $\lambda \geqq 0$.

§ 3. *The Circumscription and Inscription Theorems.*

THEOREM I. *Precisely one of a given system of ∞^2 $\alpha\beta$-arches equivalent under the group G_2 circumscribes a given $\kappa\lambda$-segment, where $\alpha > \kappa > \beta$.*

An immediately evident corollary of theorem I is

THEOREM I_0. *Precisely one of a given system of ∞^2 $(+\infty, -\infty)$-arches equivalent under the group G_2 circumscribes a given $\kappa\lambda$-segment.*

Such a system is, for instance, the system of semicircular arches with basal diameters on the x axis, and another is the system of cycloid arches with bases on the x axis.

In view of the remarks of § 2, theorem I is equivalent to

THEOREM II. *Precisely one $\kappa\lambda$-segment (κ, λ given) is inscribed in a given $\alpha\beta$-arch, where $\alpha > \kappa > \beta$.*

Since (by §1, 6°)

$$+\infty \geqq \alpha > 0 > \beta \geqq -\infty,$$

the proof of theorem II naturally falls into three parts:

The case $\kappa = 0$ is proved in § 1, 10°.

The case $\alpha > \kappa > 0$ is reduced to the preceding case $\kappa = 0$, by the transformation T_κ of §1, 9°. For (in the notations there used) the arcs AC, CB lie on opposite sides of the line AC, so that any $\kappa\lambda$-segment inscribed in the arch AB, being parallel to the line AC: $y = \kappa(x - a)$, must be inscribed in the arc AC or in the arc CB. There is none in the arc CB, since in that case each arc AC, CB would have a tangent parallel to the line AC. In the arc AC however there is precisely one; for any $\kappa\lambda$-segment P_0P_1 inscribed in the arc AC transforms by T_κ into a $\bar{\kappa}\bar{\lambda}$-segment $\bar{P}_0\bar{P}_1$ inscribed in the $\bar{\alpha}\bar{\beta}$-arch $\bar{A}\bar{C}$, where

$$\bar{\alpha} = \alpha - \kappa, \quad \bar{\beta} = \gamma - \kappa,$$

$$\bar{\kappa} = 0, \quad 2\bar{\lambda} = 2\lambda - \kappa\frac{x_1 + x_0 - 2a}{x_1 - x_0} = \frac{\bar{y}_1 + \bar{y}_0}{\bar{x}_1 - \bar{x}_0},$$

so that $\bar{\lambda} \geqq 0$, since $\bar{y}_0 \geqq 0$, $\bar{y}_1 \geqq 0$, — and conversely; and by

the preceding case there is precisely one such $\bar{\kappa}\bar{\lambda}$-segment $\bar{P}_0\bar{P}_1$ inscribed in the $\bar{\alpha}\bar{\beta}$-arch $\bar{A}\bar{C}$.

The case $0 > \kappa > \beta$ is reduced to the preceding case $\alpha > \kappa > 0$, by the transformation R of § 1, 8°. For under R a $\kappa\lambda$-segment P_0P_1 transforms into a $\bar{\kappa}\bar{\lambda}$-segment $\bar{P}_1\bar{P}_0$, and an $\alpha\beta$-arch AB transforms into an $\bar{\alpha}\bar{\beta}$-arch $\bar{B}\bar{A}$, where

$$\bar{\kappa} = -\kappa, \quad \bar{\lambda} = \lambda; \quad \bar{\alpha} = -\beta, \quad \bar{\beta} = -\alpha,$$

so that indeed $\bar{\alpha} > \bar{\kappa} > 0$.

THE UNIVERSITY OF CHICAGO,
February 6, 1904.

THE RICCATI DIFFERENTIAL EQUATIONS WHICH REPRESENT ISOTHERMAL SYSTEMS.

BY DR. EDWARD KASNER.

(Read before the American Mathematical Society, December 29, 1903.)

THE object of this note is to determine the isothermal systems of (plane) curves which can be represented by an equation of Riccati type

$$y' = P + Qy + Ry^2, \tag{1}$$

where P, Q, R denote arbitrary functions of x. The result obtained is that there are four distinct systems of this kind, namely, those given by the equations (3), (4), (6), (7) below.

It has been shown by Lie * that the differential equation of any isothermal system is of the form

$$y' = \tan F(x, y)$$

where F is a harmonic function; that is,

$$F_{xx} + F_{yy} = 0.$$

By a simple transformation this result may be expressed more conveniently for the present purpose as follows:

* Lie-Scheffers, Differentialgleichungen, 1891, p. 157.

In order that an equation $y' = f(x, y)$ *shall represent an isothermal system, it is necessary and sufficient that the function* f *satisfy the partial differential equation*

$$(2) \qquad (1 + f^2)(f_{xx} + f_{yy}) - 2f(f_x^2 + f_y^2) = 0.$$

It will be convenient to divide the discussion of the question proposed into three cases: In the first case $R = 0$ and $Q = 0$, so that the equation (1) does not involve y; in the second, $R = 0$, but $Q \neq 0$, so that the equation is linear; in the third, $R \neq 0$, so that the equation is quadratic in y.

Case I. *Equations of Type* $y' = P$.

Substituting P for f in the general condition (2), we find that P must satisfy the (ordinary) equation of second order

$$(1 + P^2)P' - 2PP'^2 = 0,$$

the solution of which is found to be

$$P = \tan(ax + b).$$

Hence the only differential equations of type I representing isothermal systems are $y' = \tan(ax + b)$.

As long as a does not vanish, this equation may be reduced, by means of a change of origin together with a change of unit distance, to $y' = \tan x$; the corresponding system of curves is

$$(3) \qquad y + \log \cos x = \text{const.}$$

When a vanishes, on the other hand, the equation may be reduced to $y' = 0$, and the corresponding system is simply a set of parallel lines

$$(4) \qquad y = \text{const.}$$

Since the equations of type I are characterized by the fact that they admit the one-parameter group of translations in the y direction, the result obtained may be stated:

If an isothermal system admits a group of translations, it may be reduced to one of the forms (4) *or* (3).

Case II. *Equations of Type* $y' = P + Qy$, *where* $Q \neq 0$.

Substituting $P + Qy$ for f in the condition (2), and equating coefficients of powers of y to zero, we find the following sys-

tem of equations for the determination of the two functions P and Q:

$$(5)\quad \begin{aligned} &Q^2Q'' - 2QQ'^2 = 0,\\ &2(QQ'' - Q'^2)P - 4QQ'P' + Q^2P'' = 0,\\ &Q''(1+P^2) + 2QPP'' - 2Q(P'^2 + Q'^2) - 4Q'PP' = 0,\\ &(1+P^2)P'' - 2P(P'^2 + Q'^2) = 0. \end{aligned}$$

By assumption, $Q \neq 0$, so that the first equation may be written $QQ'' - 2Q'^2 = 0$. The integral of this is

$$Q = \frac{1}{ax+b}.$$

Without loss of generality we may assume $Q = \alpha x^{-1}$, this being equivalent to merely a change in the axis of ordinates. Substituting this value of Q in the second equation of (5), we find

$$x^2P'' + 4xP' + 2P = 0,$$

so that P is necessarily of the form $P = Ax^{-2} + Bx^{-1}$.

The corresponding linear equation $y' = P + Qy$ is

$$y' = Ax^{-2} + Bx^{-1} + \alpha x^{-1}y = \frac{a\left(y + \frac{B}{a}\right)x + A}{x^2};$$

which by a proper change in the axis of abscissas may be reduced to

$$y' = \beta x^{-2} + \alpha x^{-1}y.$$

The corresponding values of P and Q are

$$P = \beta x^{-2},\ Q = \alpha x^{-1}.$$

Finally, the last two equations of the set (5) give the additional conditions $\beta = 0$, $\alpha = \pm 1$. The corresponding linear differential equations are $y' = y/x$ and $y' = -y/x$, the solutions of which are

$$(6)\qquad y/x = \text{const.},$$

$$(7)\qquad xy = \text{const.}$$

The only isothermal systems whose differential equation is of type II *are reducible to the systems* (6) *or* (7), *the former representing a pencil of straight lines and the latter a system of equilateral hyperbolas with common asymptotes.*

Since types I and II make up the class of linear equations:

If the differential equation of an isothermal system is linear, the system is of one of forms (3), (4), (6), (7).

Among the linear equations, the class $y' = Qy$ form a special class characterized by the property of invariance with respect to the group of affine transformations

$$x_1 = x, \quad y_1 = \lambda y. \tag{8}$$

Of the systems obtained, (4), (6) and (7) belong to this subclass.

The only isothermal systems which allow the one parameter group of affine transformations (8) *are the systems of straight lines* (4), (6) *and the system of equilateral hyperbolas* (7).

Case III. *Equations* $y' = P + Qy + Ry^2$, *where* $R \neq 0$.

In this case the discussion shows that no solutions exist. Applying condition (2), we find

$$\begin{aligned}(9)\quad &\{R''y^2 + Q''y + P'' + 2R\}\{1 + (Ry^2 + Qy + P)^2\} \\ &- 2\{Ry^2 + Qy + P\}\{(R'y^2 + Q'y + P')^2 \\ &\qquad + (2Ry + Q)^2\} = 0.\end{aligned}$$

By equating the coefficients of the various powers of y in the first member to zero, a system of seven differential equations of the second order is obtained for the determination of the three unknown functions P, Q, R. To prove that these equations, under the assumption that R does not vanish, are inconsistent, it will be sufficient to consider four of them, namely, those obtained from the sixth, fifth, fourth and second powers of y.

The first equation is

$$R^2R'' - 2RR'^2 = 0.$$

The complete solution of this equation is $(ax + b)^{-1}$, but without loss of generality we may assume

$$R = \alpha x^{-1} \quad (\alpha \neq 0). \tag{10}$$

The second equation, obtained from the coefficient of y^5 in (9), is

$$R^2Q' - 4R'R'Q + 2(RR'' - R'^2)\ Q = 0,$$

which reduces to

$$x^2\,Q'' + 4xQ' + 2Q = 0$$

after the substitution of the value of R given in (10). Integrating, we find that Q must be of the form $Ax^{-2} + Bx^{-1}$.

From the discussion up to this point, it follows that if an isothermal system of type III exists, its equation must be of the form

$$y' = \frac{\alpha xy^2 + (Bx + A)y}{x^2} + P.$$

By the substitution

$$y \sim y + \frac{B}{2\alpha},$$

equivalent to merely a change in the axis of abscissas, the equation reduces to the form

$$y' = \frac{\alpha xy^2 + \beta y}{x^2} + P$$

where β is some constant and P is some function of x (not necessarily the same as that denoted by P in the original equation). We may therefore, without loss of generality, take

$$Q = \beta x^{-2}. \tag{11}$$

The third and fourth equations referred to above, corresponding to the coefficients of y^4 and y^3 in (9), become, after reduction by means of (10) and (11),

$$x^3P'' + 4x^2P' + 2x^3P = 6\alpha x^4 + 2\beta^2\alpha^{-1}. \tag{12}$$

$$2\alpha x^{-3}(1 + P^2) + 12\beta^2x^{-6}P + (2\alpha x^{-1} + P'')(\beta^2x^{-4} + 2\alpha x^{-1}P),$$

$$-2\alpha x^{-1}(P'^2 + \beta^2x^{-4}) - 2\beta x^{-2}(4\alpha\beta x^{-3} - 4\beta x^{-3}P') \tag{13}$$

$$- 2P(4\beta^2x^{-6} + 4\alpha^2x^{-2} - 2\alpha x^{-2}P') = 0.$$

The integration of (12) gives

$$P = \alpha x + \gamma x^{-1} + \delta x^{-2} - \beta^2\alpha^{-1}x^{-3}.$$

Substituting this value in (13), we find that the coefficient of the highest power of x in the first member is $-2\alpha^3$. This however cannot vanish on account of the assumption that $\alpha \neq 0$. It follows that there are no functions satisfying condition (9) except when $R = 0$. Therefore

No equation of type III, *that is of the Riccati type proper, can represent an isothermal system.*

The complete answer to the question proposed at the beginning of the paper may now be stated:

If the differential equation of an isothermal system is included in the form $y' = P + Qy + Ry^2$, *then the system must belong to one of four species:* 1°, *a set of parallel straight lines;* 2°, *a pencil of straight lines;* 3°, *a system of equilateral hyperbolas with common asymptotes;* 4°, *a system of logarithmic cosine curves* $y + \log \cos x = \text{const.}$, *which may also be written* $e^y \cos x = \text{const.}$

COLUMBIA UNIVERSITY,
December, 1903.

ON SOME PROPERTIES OF GROUPS WHOSE ORDERS ARE POWERS OF A PRIME.

BY DR. W. B. FITE.

(Read before the American Mathematical Society, December 28, 1903.)

IN the *Transactions*, volume 3 (1902), page 334, the writer has shown that if a metabelian group of order p^m, where p is a prime, contains an abelian subgroup of order p^{m-a}, the p^a power of every operator is invariant. In the same article, page 349, it was shown that a metabelian group of odd order cannot be a group of cogredient isomorphisms if it has a set of generators such that the order of any one of them is not a divisor of the least common multiple of the orders of all the others. This latter was generalized somewhat in the BULLETIN, 2d series, volume 9 (1902), page 140. It is the purpose of the present article to carry this generalization somewhat further and to show that the first theorem is a special form of a theorem applicable to all groups whose orders are powers of a prime.

Let G be a group of order p^m and class k that contains an abelian subgroup G_1 of order p^{m-1}. If A is any operator of G_1 and B any operator of G not contained in G_1, we have

$$B^{-1}AB = At_1,$$

$$B^{-p}AB^p = At_1^p t_2^{\frac{p(p-1)}{2}} \cdots t_i^{\frac{p(p-1)\cdots(p-i+1)}{i!}} \cdots t_p = A,$$

where

$$B^{-1}t_iB = t_it_{i+1} \quad (i = 1, 2, \cdots, k-1),$$

since B^p is contained in G_1. Also

$$B^{-p}t_iB^p = t_it_{i+1}^{\frac{p(p-1)}{2}} \cdots t_{i+j}^{\frac{p(p-1)\cdots(p-j)}{(j+1)!}} \cdots t_{i+p} = t_i.$$

The exponents of the successive commutators are the binomial coefficients, and $t_j = 1$ when $j > k-1$. By giving to i in the equation above the successive values from $k-2$ to $k-p$ inclusive, we see that

$$t_j^p = 1 \ (j > k-p).$$

And in general $t_j^{p^s} = 1$ if $j > k-1-s(p-1)$. Hence $t_1^{p^r} = 1$, if $1 > k-1-r(p-1)$ or

$$r(p-1) \geqq k-1.$$

Therefore A^{p^r} is invariant in G, and the p^r power of every operator is invariant.

Conversely, if $s(p-1) < k-1$, the p^s power of every operator is not invariant. For otherwise

$$t_i^{p^{s-j}} = 1,$$

if

$$j(p-1) < i \quad (j = 0, 1, 2, \cdots, s).$$

From this it follows that $t_i = 1$, if $s(p-1) < i$. Hence $t_{k-1} = 1$, and t_{k-2} is invariant in G. That is, the $(k-2)$nd adjoined group is abelian, and G is of class l, where $l < k$. The results thus obtained can be formulated into the

THEOREM: *If a group G of order p^m (p a prime) and class k contains an abelian subgroup of order p^{m-1}, the p^r power of every operator is invariant if $r(p-1) \geqq k-1$. Conversely, if G contains an abelian subgroup of order p^{m-1} and if the p^r power of every operator is invariant, G is of class k, where $k-1 \leqq r(p-1)$.*

We concern ourselves now with the case in which the abelian subgroup is of order p^{m-a}. Let this subgroup be denoted by G_1 and let G_2 be a subgroup of G of order p^{m-a+1} that contains

G_1, and G_2 a subgroup of order p^{m-s+2} that contains G_2. If B is an operator of G_2 that is not in G_2, $B^{p^{r+1}}$ is invariant in G_2. If A is any operator of G_2 that is not in G_1, A^p is invariant in G_2. Now

$$B^{-1}A^pB = A^pt_1,$$

$$B^{-p}A^pB^p = A^pt_1^p \cdots t_i^{\frac{p(p-1)\cdots(p-i+1)}{i!}} \cdots t_p = A^p,$$

where

$$B^{-1}t_iB = t_it_{i+1} \qquad (i = 1, 2, \cdots, k-1).$$

The commutators t_i are invariant in G_2 since A^p is.

As before, it follows that $t_1^{p^r} = 1$, where $r(p-1) \geqq k-1$. Hence

$$B^{-1}A^{p^{r+1}}B = A^{p^{r+1}}t_1^{p^r} = A^{p^{r+1}},$$

and $A^{p^{r+1}}$ is invariant in G_2. If C is any operator of G_1, then C^{p^r} is invariant in G_2, and hence $C^{p^{2r}}$ is invariant in G_2.

We have therefore shown that the p^{2r} power of every operator of G_2 is invariant. Now by induction and a repetition of the argument just used we reach the

THEOREM: *If a group G of order p^m and class k contains an abelian subgroup of order p^{m-s}, the p^{rs} power of every operator is invariant, where $r(p-1) \geqq k-1$.*

Returning now to the case in which the abelian subgroup is of order p^{m-1}, we see that if G_1 is of type $(m_1, m_2, \cdots, m_n)$, where l of the quantities m_i are equal to 1, then $k \leqq l+2$ or $> p$. For the commutators of G are all contained in G_1 and if $k \leqq p$ they are all of order p. Furthermore, the pth power of every operator is invariant. Hence the order of the commutator subgroup of G', the group of cogredient isomorphisms of G, is less than, or equal to, p^l, and G' is of class k', where $k' \leqq l+1$.* Therefore $k \leqq l+2$. In particular, if $l = 0$, either G is metabelian, or $k > p$.

Suppose now that G is of order p^m and class k and that B' is the operator of G' that corresponds to B of G.† If B' is of order p^β and A is any operator of G, we have, if the commutator subgroup of G is abelian,

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* *Transactions*, vol. 3 (1902), p. 350.

† *Loc. cit.*, p. 332.

$$B^{-p^\beta}AB^{p^\beta}$$

$$= At_1^{p^\beta}\cdots t_j^{p^\beta(p^\beta-1)\cdots(p^\beta-j+1)/j!}\cdots t_{k-1}^{p^\beta(p^\beta-1)\cdots(p^\beta-k+2)/(k-1)!}$$

$$= A,$$

$$B^{-p^\beta}t_iB^{p^\beta} = t_it_{i+1}^{p^\beta}\cdots t_{k-1}^{p^\beta(p^\beta-1)\cdots(p^\beta-k+i+2)/(k-1)!} = t_i,$$

$$(i = 1, 2, \cdots, k-2).$$

Since $t_i^{p^{\beta+j}} = 1$, when $(j+1)(p-1) \equiv k-i$, it follows that $t_1^{p^{\beta+r}} = 1$ when $(r+1)(p-1) \equiv k-1$.

In a similar way it can be seen that

$$t_1^{p^{\alpha+r}} = 1,$$

where p^α is the order of A'. Hence $t_i^{p^{\alpha+r}} = 1$, since

$$B^{-1}t_{i-1}^{p^{\alpha+r}}B = t_i^{p^{\alpha+r}}t_{i-1}^{p^{\alpha+r}} = t_{i-1}^{p^{\alpha+r}} \qquad (i = 2, 3, \cdots, k-1),$$

when

$$t_{i-1}^{p^{\alpha+r}} = 1.$$

But we have just seen that

$$t_1^{p^{\alpha+r}} = 1.$$

Suppose now that $k-s$ is the greatest multiple of $p-1$ less than $k-1$. Then since $G^{(s-1)}$, the $(s-1)$th adjoined group of G, is of class $k-s+1$ and $A_{(s-1)}^{p^\alpha}$ is commutative with every operator of $G^{(s-1)}$, we have $t_1^{p^{\alpha+r'}(s-1)} = 1$, where

$$(r'+1)(p-1) = k-s.$$

Obviously $r' < r$. Now

$$B^{-1(s-2)}t_1^{(s-2}B^{(s-2)} = t_1^{(s-2)}t_2^{(s-2)}.$$

But since $t_1^{p^{\alpha+r'}(s-1)} = 1$, we have $t_2^{p^{\alpha+r'}(s-2)} = 1$.

By continuing this reasoning we get $t_s^{p^{\alpha+r'}} = 1$. Hence $t_i^{p^{\alpha+r'}} = 1$, when $i > s$. More generally, we have in $G^{\{s+l(p-1)-1\}}$ the invariant operator

$$A^{p^\alpha}\{s+l(p-1)-1\}.$$

Hence

$$t_1^{p^{a+r_1}\{s+l(p-1)-1\}} = 1,$$

where $r_1 = r' - l$. Then $t_i^{p^{a+r_1}} = 1$, when $i = s + l(p-1)$.

This is equivalent to the condition

$$(r_1 + 1)(p-1) \geqq k - i.$$

We conclude from this that

$$B^{-p^{a+r}} A B^{p^{a+r}} = A.$$

Therefore if A' is of order p^a, the p^{a+r} power of every operator of G is commutative with A, when

$$(r+1)(p-1) \geqq k - 1.$$

From this result follows the

THEOREM: *If a group of order p^m and class k has an abelian commutator subgroup, its group of cogredient isomorphisms cannot have a set of generators such that the order of any one of them is greater than p^r times the order of every one of the others, r being defined by the relation $(r+1)(p-1) \geqq k-1$.*

We add a formula for the nth power of the product of any two operators of a group of the 3d class. Let A and B be these operators. Then if

$$B^{-1}AB = At_1, \quad B^{-1}t_1B = t_1t_2, \quad A^{-1}t_1A = t_1h,$$

we have

$$(BA)^n = B^nA^nt_1^{\frac{n(n-1)}{2!}} t_2^{\frac{n(n-1)(n-2)}{3!}} h^{\frac{n(n-1)(2n-1)}{3!}}.$$

CORNELL UNIVERSITY,
December, 1903.

WHITTAKER'S MODERN ANALYSIS.

A Course of Modern Analysis. By E. T. WHITTAKER. Cambridge, England, University Press, 1902. xvi+ 378 pp.

This book, as the subtitle explains, is intended as "an introduction to the general theory of infinite series and of analytic functions; with an account of the principal transcendental functions." There is certainly room for a book of this sort, especially as the transcendental functions treated are of the most varied kinds, including the gamma function, hypergeometric functions with the special and limiting cases of Legendre's and Bessel's functions, as well as elliptic functions, both Weierstrass's and Jacobi's normal forms being treated in detail. Somewhat more than half the book is devoted to an account of these functions. Part I, which to some extent may be regarded as introductory to this second part, is designed, to use the author's own words, to contain "an account of those methods and processes of higher mathematical analysis which seem to be of greatest importance at the present time." There is so much of interest and importance in these chapters, parts of which can be found nowhere else in the English language, that it may seem ungracious to criticise the choice of material which has commended itself to the author. There are, however, one or two matters of fundamental importance whose omission it is hard to justify. The subject of series, which claims the lion's share, forms only one of several infinite processes which occur constantly in analysis and which are, in fact, freely employed in the latter part of the book, though usually without justification except in the case of series. As an illustration of what is meant, attention may be called to the treatment of infinite products. Four pages are devoted to the question of the convergence and absolute convergence of such products, but the conception of their uniform convergence is not touched upon. Or again, to come to a matter of still greater importance, the distinction between finite and infinite definite integrals * is not

*I follow T. J. I'A. Bromwich (*Proc. London Math. Soc.* Ser. 2, vol. 1, p. 176 where reference to G. H. Hardy is made) in translating the German terms *eigentliches Integral* and *uneigentliches Integral* by *finite integral* and *infinite integral* respectively. The terms *proper* and *improper* integrals, which have sometimes been used, although they come nearer to the German, do not seem

brought out, nor is it explained what precautions must be used in working with infinite integrals. Thus, for example, the question as to whether we have the right to assume that

$$\frac{d}{dx}\int_c^\infty \phi(t,x)dt = \int_c^\infty \frac{\partial\phi(t,x)}{\partial x}\,dt,$$

is a question precisely on a par, both in its internal nature and in its practical importance, with the question whether we have the right to differentiate a series term by term. Mr. Whittaker considers this latter question with care; the former he does not even mention, although he frequently has occasion to use infinite integrals in the latter part of his book. These three subjects — series, infinite products, and infinite integrals — are merely special cases, and by no means the only important special cases, of an infinite process of the sort indicated by the equation

$$F(x) = \lim_{\alpha=k} f(x,\alpha)$$

where k may be finite or infinite, and α may or may not be restricted to a discrete set of values in approaching k. Thus we have the case of a series if in the above formula we let $k = +\infty$, restrict α to positive integral values, and let $f(x, \alpha)$ denote the sum of the first α terms of the series. Or, again, we obtain the case of an infinite integral by letting

$$f(x, \alpha) = \int_c^\alpha \phi(t, x)dx, \qquad k = +\infty.$$

Now a fundamental question is whether $f(x, \alpha)$ approaches its limit $F(x)$ *uniformly* throughout a certain interval, that is whether, after choosing our limit of error ϵ small at pleasure, we can then push our limiting process so far that, from this point on, $f(x, \alpha)$ differs from its limit $F(x)$ at all points of the interval by less than ϵ. If this is the case, we say that our infinite process converges uniformly, whether this process be expressed

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desirable from the point of view of pure English. It will be noticed that the terms finite and infinite, as here used, correspond precisely to the use of these words in finite and infinite products, finite sums and infinite series, etc. *i. e.*, the infinite integral is the limit of a finite integral. The term infinite integral is intended to apply not merely to the case mentioned in the text in which one of the limits of integration is infinite, but also to the case in which the function becomes infinite at one end of the interval.

as a series, as an infinite product,* or as an infinite integral. When once the idea of the uniform convergence of series has been grasped and its applications understood, it would be a matter of only a very few pages to treat the subject from the much more general standpoint just indicated.

Before leaving this subject of uniform convergence we may perhaps be permitted to express the opinion that, especially in a treatise devoted largely to analytic functions of a complex variable, Weierstrass's theorem should not have been omitted which states † that if the terms of a series are functions of a complex variable which throughout a certain two-dimensional region are analytic, and if throughout this region the series converges uniformly, it represents an analytic function whose derivatives of all orders can be found by differentiating the series term by term. This theorem is, in fact, tacitly assumed in the second part of the book.

If we have thus indicated a number of unfortunate omissions, it is only right to add that other important matters of great interest are included which have not yet made their way into the ordinary treatises. Among these may be particularly mentioned infinite determinants and asymptotic developments.

Turning now to the second part of the book in which the more important transcendental functions are treated, we find a large amount of detailed information which cannot fail to prove valuable to the student who wishes to penetrate into the theory of one or the other of these functions. Although general points of view are only lightly touched upon in these chapters they are by no means omitted. The author might, however, have made his treatise more valuable by going a little farther in this direction. Thus, to mention a small matter, Gauss's formula for expressing the logarithmic derivative of the gamma function as a definite integral (p. 186) really holds a unique position among the numerous definite integral formulæ connected with the gamma function in that the integrand is single valued instead of being infinitely multiple valued; an advantage well known to Gauss, as is evident from his remarks in a letter to Bessel in an analogous case. A line or two of text would have made this fact clear. Again, Weierstrass's

* A slight deviation of phraseology is probably desirable for infinite products so that such a product which at one or more points of the interval "diverges" to the value zero shall not be called uniformly convergent.

† This theorem can be at once extended to other infinite processes.

theorem that the gamma function satisfies no algebraic differential equation might well have been mentioned, with a reference to Hölder's proof, as this theorem gives the student a much clearer idea of the place among the other transcendental functions which this function occupies. We mention these two points merely as samples of the way in which by an incidental remark now and then the reader's horizon might be enlarged.

While the book before us is still far from attaining the standard of rigor which French, German, and Italian writers now regard as almost a matter of course, it is a gratifying sign of progress to find in an English book such an attempt at rigorous treatment as is here made. It is only fair to the reader, however, to warn him that he should be on the lookout for pitfalls, for they abound in this subject, and the author has not always succeeded in avoiding them.

The examples which Mr. Whittaker, following the custom of English writers, has introduced in large number into this treatise cannot fail to be of great service in making the subject a living one to many readers, or rather students, of his book. The solution (by the student himself be it understood, not, after the approved German fashion, by the writer of the treatise) of a problem, well chosen to illustrate a theoretical point, often does more to clarify the theory and impress it on the student's mind than any amount of direct study of the principle concerned. While the advantage of such problem solving decreases as the student becomes more mature, it by no means disappears even in the case of most full fledged mathematicians; and many a person well beyond his student years will find that he can spend profitable hours in sharpening his wits on the whetstone with which he is here supplied in the shape of problems.

MAXIME BÔCHER.

HARVARD UNIVERSITY,
January, 1904.

ENRIQUES'S PROJECTIVE GEOMETRY.

Vorlesungen über projective Geometrie, von FEDERIGO ENRIQUES, ordentlicher Professor an der Universität Bologna. Deutsche Ausgabe von Dr. HERMANN FLEISCHER, mit einem Einführungswort von FELIX KLEIN. Leipzig, Teubner, 1903. xiv + 374 pp., 187 figs. Price 8 marks.

ITALY has been the home of projective geometry for the last twenty years. In 1894, when Professor Enriques was called to the University of Bologna as professor of projective geometry, he set himself the problem of preparing a systematic treatise on the subject, based on a set of independent axioms. The Lezioni di geometria proiettiva was published at Bologna, in 1898. The next year Hilbert's Foundations of Geometry appeared, at which time Professor Klein expressed the wish for a translation of Enriques's book, in order to have the systematic treatment of projective geometry as well as of metric geometry accessible to German students. The German translation differs from the original only in the correction of various little slips and omissions, the amplification of the appendix, and in the presence of an index. In two respects the make up of the book is hardly up to the Teubner standard; spaced letters are used instead of italics for emphasis, and the figures are rather crude; in other respects the appearance of the book is very presentable.

The axioms, six in number, are the following:

First group, axioms of connection:

I. In a three-dimensional configuration, any two elements determine a one-dimensional configuration to which they belong and which is contained in the given configuration.

II. In a three-dimensional configuration, any three elements not contained in a one-dimensional configuration, determine a two-dimensional configuration to which they belong and which is contained in the given configuration.

III. In a three-dimensional configuration, an element and a configuration of one dimension determine, when the former is not contained in the latter, a two-dimensional configuration to which they belong.

Second group, axioms of order:

IV. The elements of a configuration of one dimension can be regarded as arranged in a natural cyclic order in either sense.

V. If two configurations of one dimension are perspective and an element describes a segment upon one, then the corresponding element will describe a segment on the other.

VI. Dedekind's axiom of continuity.

Numerous illustrations are given to make the meaning of these axioms clear. The advantage of keeping them in the abstract form is seen in their immediate application to dual theorems. The principle of duality is difficult to many students simply because their previous training in analytic geometry has all been on one side. The usual von Staudt construction is given to prove the invariance of the fourth harmonic element, then various metric illustrations are added, each one being marked by a star. This is a decided improvement over von Staudt's arrangement. In the classic Geometrie der Lage all proofs are established by projective principles, but metric illustrations are brought in without warning, which confuses the reader. The present book is, I believe, unique in its direct deduction of each metric theorem as a particular case of a projective theorem. In all other works known to me a few metric properties are established directly, then others are founded on these, no further use being made of projective methods.

The principles thus far established make it possible to prove rigorously the fundamental theorem, after which the idea of the projective group is introduced. This procedure seems for more natural than that employed in the well known books of Reye and of Cremona. The unfortunate place that involution occupies in Reye's book is perhaps the cause of its being frequently slighted by students. In the present work involution comes immediately and naturally after the chapter on projectivity of one-dimensional forms. Chapter 7 enables the author to give a logical purely geometric definition to euclidean geometry. If besides axioms I–VI one assumes the existence of an elliptic involution, having the circular points as double elements, all the theorems of metric geometry can be established. (This is equivalent to having given two right angles with common vertex and common plane.)

Chapter 8 is devoted to projectivity between two forms of two dimensions, with a full discussion of homology and polarity, the latter being defined as follows: a reciprocity (duality) in the plane is a polarity when a triangle exists in which each

vertex corresponds to the opposite side. Thus far 186 pages have been covered without mentioning any locus or envelope, but nowhere does the presentation seem forced or dry. In particular, the systematic development of each successive topic from axioms I to VI is rigidly adhered to. Whether students would find this treatment as interesting as that found in most other texts I hardly know, but I believe that those not spoiled by our present excessive instruction in analytic geometry of the point will find it not only profitable but thoroughly enjoyable reading. A conic is now defined as the locus of a point which lies on its polar (or as envelope of a line which passes through its pole), and by means of this definition the usual theory of pole and polar with regard to a conic is easily established. It is readily proved that the conic is also the locus of intersection of corresponding elements of two elementary forms of one dimension. From this standpoint it is evident that the conic is self-dual. The theorems of Pascal and Desargues, together with the usual deductions, are proved and discussed in 12 pages. One objection to this chapter is its rather limited adaptability to problems in construction. A student readily grasps the idea of Steiner's construction and can quickly get a fair idea of the appearance of a locus or an envelope. What the von Staudt method needs is a suitable list of easy exercises and applications.

Chapter 10, projectivity between conics, can be compared with parts of Chapter 11 in Reye's first volume; the present treatment is more general, since it shows that any pair of real elements can be replaced by a suitable pair of conjugate imaginary elements. Added to the usual list of problems of the second degree is a good discussion of Steiner's theorem that every such problem can be solved by means of the ruler and one fixed circle, and the further theorem that all such problems may also be solved by means of a ruler with two parallel edges. Several pages are given to the rigorous discussion of problems of the third degree, one of those treated being the determination of the double elements of two superposed projective spaces of two dimensions. Chapters 12 and 13 are concerned with foci, centers, diameters, and axes of conics. By means of the absolute involution on the infinite line it follows immediately that a point (focus) such that conjugate lines through it are at right angles is necessarily the intersection of isotropic tangents to the conic. The method of proof is everywhere more direct

than in the older texts, and by means of duality in three dimensions it is shown that many theorems can be applied directly to quadric cones. The chapter on the axes of quadric cones is similar in mode of treatment to the corresponding chapter in Reye.

Finally, the last chapter is devoted to projective spaces of three dimensions. Two proofs are given of the theorem that five pairs of corresponding elements are sufficient to determine the projectivity. Homology or central (space) perspective, and axial involution are treated at some length. Direct and inverse congruence are shown to be particular cases of homology, the latter being harmonic. An appendix treats of cyclic collineations and the various finite groups defined by them; two pages are given to abstract geometry in which it is shown that no space intuition is needed to develop projective geometry from axioms I to VI; an outline of transformations of space by which spheres are transformed into spheres is followed by a discussion of projective coördinates, imaginary elements and a historical sketch.

It is curious that no mention is made of the hyperboloid except incidentally in connection with the fundamental theorem. In general, with respect to the applicability of the preceding principles to higher configurations, no part of the book is so suggestive as Chapter 11 of Reye, which has furnished the incentive to many a memoir. On the other hand, what one does acquire is a ready confidence in the truth or falsity of a theorem concerning geometric elements. One is not taught to work the machine, but rather to think geometrically. Possibly more time might be required to complete the text than is usually devoted to the subject, but the student who masters it will be better prepared to understand other transformations and the training acquired will be in harmony with his subsequent studies in geometry.

VIRGIL SNYDER.

CORNELL UNIVERSITY,
January 12, 1904.

SHORTER NOTICES.

Leçons de Mécanique Elémentaire. Par P. APPELL et J. CHAPPUIS. Paris, Gauthier-Villars, 1903. viii + 177 pp.

MATHEMATICAL instruction in the secondary schools of France has recently been passing through a period of transition. With the object of making the matter taught less abstract the study of kinematics and mechanics has been introduced into those schools, and the book under review is prepared especially to meet the new demands on the students. The title is, however, misleading, as the book deals with kinematics only; a second volume, devoted to the needs of la classe de mathématiques, will follow.

The presentation given in the volume at hand is detailed and thorough; in fact, so carefully are the ideas of segments, vectors and projections developed that it is with something of a shock that one comes, in theorem II, page 18, to the as yet undefined term resultant. The method is geometric rather than analytic. Vectors are made of fundamental importance, as is evidenced by the fact that velocity and acceleration are defined to be certain vectors; and the definition and notation of the derivative are used frequently. There are numerous striking illustrations and numerical examples interspersed in the text, but no problems for the student are given.

The first chapter, comprising pages 1–52, contains an admirable elementary treatment of segments on an oriented line straight or curved, vector addition, and the theory of projection and of the moment of a vector with respect to a point, an axis, or a second vector.

The second chapter is devoted to kinematics and falls into four parts. In the first part, pages 53–73, the ideas of motion and time are considered. The idea of motion is, very sensibly, tacitly assumed without even mention of the impossibility of defining motion; that motion is purely relative is, however, very carefully brought out by numerous examples. The ideas of time and its measurement are considered more in detail, and the mechanical and electrical devices connected with the pendulums in laboratory use are explained.

The second part of the chapter, pages 74–118, deals with the kinematics of a point. Uniform motion and uniformly accel-

erated motion along a straight line are considered in detail but it is to be remarked that no reference is made to freely falling bodies. The ideas of the velocity and acceleration of any moving point are carefully explained and application is made to motion in a circle.

The study of translations, rotations, and screw motions occupies the third part, pages 119–154. The consideration of a number of machines and mechanical devices in which these motions occur shows the practical trend of the book.

Finally, in pages 155–171, the motion of a body relative to a system which is itself in motion is considered.

ARTHUR SULLIVAN GALE.

Annuaire pour l'An 1904, publié par le Bureau des Longitudes. Paris, Gauthier-Villars.

IN accordance with the plan foreshadowed in last year's issue, the Annuaire for 1904 does not contain any geographical or statistical details. Nor does it give astronomical information of "constant" value; only that part which varies from year to year is retained. These omitted portions are to be present in the volume for 1905, while certain parts which appear now will not be given then.

On the other hand, the information on the parts retained has been largely increased, so much so that, in spite of the omissions, the volume, apart from the appendix which consists of special articles, shows an increase of 64 pages. Most of the sections have been rewritten with additional tables, and the constants have, as far as possible, been brought up to date. A reference to the preface — too long to insert here — will show the directions in which alterations have occurred.

The notices are two: "Note sur la conférence internationale tenue à Copenhague en août 1903," by M. Bouquet de la Grye, which gives a full account of the work done at the meeting, and "Explication élémentaire des marées," by M. P. Hatt. In the latter, the writer attempts to set forth in an elementary way the various forces which produce the tides. He makes a careful distinction between the equilibrium theory and the dynamical theory and in general explains them more fully than Professor Darwin in his well-known work. M. Hatt, however, has hardly achieved the same success; even to one moderately familiar with the problem, his article is not abso-

lutely easy reading. Nevertheless, it is to be welcomed as another excellent description, without complicated mathematics, of the forces which produce tidal phenomena and of the way in which those phenomena are produced.

ERNEST W. BROWN.

ACKNOWLEDGMENT.

My attention has been called to a paper by Darboux, page 55, volume 17, of the *Mathematische Annalen*, in which he proves that a one-to-one point transformation of all points of the projective plane (or of projective space) which carries over collinear points into collinear points is a projective transformation. So much of my paper in the February number of the BULLETIN as refers to the fact that it is unnecessary to require that the transformation be continuous leads therefore to no new result, although the method used is different from Darboux's. I mention in passing that all my results admit of easy generalization to space of three dimensions.

Near the close of his article Darboux shows that, also in the case of a circular transformation, it is not necessary to require explicitly that the transformation be continuous. The method which he uses applies equally well to the more general case in which the transformation is not defined for all points of the plane and leads at once, if we make use of the theorem I proved in the article referred to above, to the theorem:

Suppose we have a one-to-one correspondence between the points of two point sets S and S' (in the plane or in space) each of which has an interior point, such that four concyclic points in either set have concyclic images. Then the transformation of S into S' can be generated by the combination of inversions, including reflections on lines or planes, and rigid motions.

ELIJAH SWIFT.

ERRATA.

The attention of the editors has been called to the following errata occurring in the present volume of the BULLETIN:

Page 134. The sentence beginning at the foot of the page requires an obvious modification for the case where N is of the form $4k + 1$.

Page 183, line 22, at the end, for (13) (24) read: (13), (24).

Page 238, lines 38 and 40, for $u = gt$, $a \cdot b$, and $a \times b$ read: $\mathbf{v} = \mathbf{g}t$, $\mathbf{a} \cdot \mathbf{b}$, and $\mathbf{a} \times \mathbf{b}$.

Page 259, line 22, for $f(x)$ read: the coefficients of $f(x)$.

NOTES.

THE fifth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY will be held at Stanford University on Saturday, April 30, 1904. Titles and abstracts of papers to be presented at this meeting should be in the hands of the Secretary of the Section, Professor G. A. MILLER, Stanford University, not later than April 16.

THE April number (volume 5, number 2) of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "An existence theorem for a differential equation of the second order, with an application to the calculus of variations," by G. A. BLISS; "Determination of all the subgroups of the known simple group of order 25920," by L. E. DICKSON; "On the invariants of quadratic differential forms, II," by C. N. HASKINS; "On the coefficients in the product of an alternant and a symmetric function," by E. D. ROE, JR.; "The groups of order p^3q^3," by F. N. COLE; "Green's theorem aud Green's functions for certain systems of differential equations," by C. M. MASON; "Studies in the general theory of ruled surfaces," by E. J. WILCZYNSKI.

THE January number (volume 5, number 2) of the *Annals of Mathematics* contains the following papers: "Note on a series of analytic functions," by C. ARZELÀ; "Graphs of the functions Π and Ψ," by E. P. R. DUVAL; Examples of non-applicable surfaces having the same gaussian curvature at corresponding points," by A. S. GALE; "The mathematical theory

of the top, II," by A. G. GREENHILL; "A relation between the circular and the projective transformations of the plane," by EDWARD KASNER.

The delay in the appearance of the number, for which the editors desire to express their regret, has been due to labor conditions.

AT the meeting of the London mathematical society held on February 11, 1904, the following papers were read: By MR. G. H. HARDY, "On the roots of the equation $1/(x+1)! = c$"; by Professor A. C. DIXON, "On a certain double integral"; by Professor A. W. CONWAY, "On an appropriate form of conductor for a moving point singularity"; by Professor H. LAMB, "On group velocity"; by Dr. E. W. HOBSON, "On point-wise discontinuous functions of a real variable"; by Mr. G. H. HARDY, "Some extensions of Abel's theorem on power series on the circle of convergence."

THE second international congress of philosophy will be held at Geneva, Switzerland, September 4–8, 1904. The international congress of the history of sciences has been incorporated into the congress of philosophy as a section. Professors D. E. SMITH and F. CAJORI are the American members of the international commission of the congress.

THE French association for the advancement of science will hold its next meeting on August 4, 1904, at Grenoble, under the presidency of C. A. LAISANT.

THE general division of the Technical high school in Munich announces the following prize problem for the present year:

"The representation of plane curves by means of their intrinsic equations has recently received considerable attention. Professor G. SCHEFFERS has defined as intrinsic equations of a surface those which express a relation between the derivatives of the principle radii of curvature with respect to the arcs of the corresponding lines of curvature and the radii themselves. The explicit derivation of these equations for the general quadric surface is desired, and the deduction from them of the principal characteristics of the surfaces."

Competing memoirs must be in the hands of the secretary by October 15, 1904.

AN English translation of Goursat's Cours d'analyse mathématique is being prepared by Professor E. R. HEDRICK, with

the assistance of the author and Professor W. F. OSGOOD. While following the French text, the English translation will include a complete revision of the whole work. It will be published in a few weeks by Ginn and Co.

THE universities below offer advanced mathematical courses during the summer session of 1904, as follows:

CORNELL UNIVERSITY. — By Professor L. A. WAIT: Advanced analytic geometry, three hours; calculus, two nours. — By Professor G. W. JONES: Higher algebra, five hours. — By Professor J. I. HUTCHINSON: Integral calculus, two hours; theory of functions of a complex variable, three hours. — By Dr. W. B. FITE: Differential equations, five hours; Number theory, two hours; Theory of groups, three hours.

UNIVERSITY OF PENNSYLVANIA. — By Professor E. S. CRAWLEY: Modern analytic geometry. — By Professor G. E. FISHER: Theory of functions of a complex variable. — By Professor I. J. SCHWATT: Theory of functions of a real variable. — By Dr. F. H. SAFFORD: Differential equations.

THE several institutions below offer during the second semester of the present academic year courses in mathematics as follows:

COLLÈGE DE FRANCE. — By Professor C. JORDAN: Construction of solvable groups, two hours. — By Professor J. HADAMARD: Equations of elasticity, two hours. — By Professor C. BAIRE: Relation between the ideas of continuity and discontinuity in certain questions of analysis, three hours.

UNIVERSITY OF HALLE. — By Professor G. CANTOR: Theory of numbers, five hours; Seminar, two hours. — By Professor A. WANGERIN: Selected chapters from the theory of potential, one hour; General theory of twisted curves and surfaces, five hours; Differential calculus with exercises, five hours; Seminar, two hours. — By Professor V. EBERHARD: Analytic geometry of the plane and introduction to that of space, four hours; with exercises, one hour. — By Professor H. GRASSMANN: Analytic mechanics III, three hours; with exercises, one hour. — By Dr. F. BERNSTEIN: Theory of functions, four hours; with exercises, one hour.

UNIVERSITY OF JENA. — By Professor A. GUTZMER: Differential calculus, with exercises, five hours: Calculation of potential, four hours; Mathematical colloquium, two hours. — By

Professor J. THOMAE: Elliptic functions, four hours; Projective geometry, two hours. — By Professor G. FREGE: Analytic geometry, four hours; with exercises, two hours.

UNIVERSITY OF PARIS. — By Professor G. DARBOUX: Higher geometry, two hours. — By Professor E. GOURSAT: Ordinary and partial differential equations, two hours. — By Professor P. PAINLEVÉ: Rational mechanics II, two hours. — By Professor G. KOENIGS: Principles of elasticity, two hours. — By Professor E. Picard: Theory of functions of several variables. — By Professor J. BOUSSINESQ: Equilibrium of the elasticity of the sphere. Mathematical conferences will be held as follows: By Professor S. RAFFY: Higher geometry and the calculus. — By Professor P. PUISEUX: Mechanics. — By Professor H. ANDOYER and DR. BLUTEL: General conferences for students of mathematics.

UNIVERSITY OF ROSTOCK: — By Professor O. STAUDE: Analytic mechanics, four hours; analytic geometry of space, four hours.

DURING the academic year from April 1, 1902 to March 31, 1903, the German universities conferred 27 doctorates in pure and applied mathematics, two successful candidates being Americans.

ON November 29, 1903, occurred the centenary of the birth of CHRISTIAN DOPPLER, the Austrian mathematician and physicist. A memorial tablet was placed upon the house in which he was born, at Salzburg.

PROFESSOR D. RUDEL, of Nürnberg, has been awarded the silver medal of the Bavarian academy of sciences for his investigations in climatology.

PROFESSOR C. NEUMANN, of Leipzig, has been awarded the Maximilian order by the Bavarian government.

PROFESSOR L. KIEPERT, rector of the technical high school in Hanover, has been awarded the order of the red eagle of the third class by the German emperor.

THE Prussian academy of sciences has voted a grant of 250 marks to Professor H. A. SCHWARZ to enable him to prepare a catalogue of the literature of minimal surfaces.

PROFESSOR E. STUDY, of Greifswald, has been appointed professor of mathematics in the University of Bonn, as successor of the late Professor R. LIPSCHITZ.

DR. R. DAUBLEBSKY v. STERNECK has been appointed acting professor of mathematics in the University of Czernowitz.

DR. FR. KREUTZBERG has been appointed professor of applied mathematics in the Royal Academy, at Posen.

PROFESSOR D. E. SMITH has been made a member of the comité de patronage of *L'Enseignement mathématique*, representing with Professor ALEXANDER ZIWET the United States upon that journal.

PROFESSOR L. E. DICKSON has been appointed research assistant of the Carnegie Institution, and will have a year's leave of absence from his duties at the University of Chicago.

DR. C. N. HASKINS, instructor in mathematics in the Sheffield Scientific School of Yale University, has resigned his position on account of poor health.

MR. J. F. TRAVIS has been appointed instructor in mathematics in the Iowa State College, at Ames, Iowa.

Professor WILHELM SCHELL died at Karlsruhe, Baden, on the thirteenth of February, 1904. His principal scientific work is the well-known treatise on mechanics which under the title "Theorie der Bewegung und der Kräfte" (Leipzig, Teubner) first appeared in 1870 in a single volume and was enlarged to two volumes in 1879–80. This work, which owing to its extensive and accurate bibliographical indications forms a most valuable book of reference, is especially noteworthy for the stress laid upon geometric methods and the attempt to apply to mechanics certain ideas of synthetic geometry. Professor Schell is also the author of a valuable work on twisted curves, "Allgemeine Theorie der Curven doppelter Krümmung in rein geometrischer Darstellung" (Leipzig, Teubner 1859, enlarged 1898), and of several articles published in the *Zeitschrift für Mathematik und Physik*.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AUTONNE (L.). Sur la décomposition d'une substitution linéaire, réelle et orthogonale, en un produit d'inversions. Paris, 1903. 8vo. 130 pp. Fr. 6.00

BALL (W. W. R.). Breve compendio di storia delle matematiche. Versione dall'inglese con note, aggiunte e modificazioni di D. Gambioli e G. Puliti, riveduta e corretta da G. Loria. Vol. II: Le matematiche moderne sino ad oggi. Bologna, Zanichelli, 1904. 8vo.

BIANCHI (L.). Geometria analitica. Pisa, 1903. 8vo. Fr. 15.00

BREWSTER (H. B.). Collineations of space which leave invariant a quadratic surface. (*Kansas University Science Bulletin*, Vol. 1 (1902), pp. 281–303.) Lawrence, Kan., 1902. 8vo.

BURALI-FORTI (C.). Lezioni di geometria metrico-proiettiva. Torino, Bocca, 1904. 8vo. 12 + 308 pp. (Biblioteca matematica, Vol. X.) Fr. 8.00

DÖLP (H.). Aufgaben zur Differential- und Integralrechnung, nebst den Resultaten und den zur Lösung nötigen theoretischen Erläuterungen. Neu bearbeitet von E. Netto. 10te Auflage. Giessen, Ricker, 1903. 8vo. 3 + 216 pp. Cloth. M. 4.00

——. Die Determinanten, nebst Anwendung auf die Lösung algebraischer und analytisch-geometrischer Aufgaben; elementar behandelt. 6te Auflage. Darmstadt, Roether, 1903. 8vo. 4 + 95 pp. M. 2.00

FRENET (F.). Recueil d'exercices sur le calcul infinitésimal, à l'usage des candidats à l'Ecole polytechnique et à l'Ecole normale, des élèves de ces écoles et des aspirants à la licence ès sciences mathématiques. 6e édition, avec un appendice sur les résidus, les fonctions elliptiques, les équations aux dérivées partielles, les équations aux différentielles totales, par H. Laurent. Paris, Gauthier-Villars, 1904. 8vo. 45 + 538 pp. Fr. 8.00

GAMBIOLI (D.). See BALL (W. W. R.).

HASKELL (M. W.). Generalization of a fundamental theorem in the geometry of the triangle. (*Archiv der Mathematik und Physik*, Series 3, Vol. 5, pp. 278–281.) Leipzig, Teubner, 1903. 8vo.

JAHRBUCH über die Fortschritte der Mathematik. Begründet von C. Ohrtmann, herausgegeben von E. Lampe und G. Wallenberg. Vol. 32: Jahrgang 1901. Heft 2. Berlin, Reimer, 1904. 8vo. M. 6.60

KEYSER (C. J.). Concerning the concept and existence-proofs of the infinite. (*The Journal of Philosophy, Psychology, and Scientific Methods*, Vol. 1, pp. 29–36.) New York, 1904. 8vo.

KIRCHBERGER (P.). Ueber Tchébychefsche Annäherungsmethoden. (Diss.) Göttingen, 1902. 8vo. 96 pp.

LAMPE (E.). See JAHRBUCH.

LAURENT (H.). See FRENET (F.).

LORIA (G.). See BALL (W. W. R.).

MAGNANI (F.). La funzione del Green per un sistema di sfere esterne le une alle altre. Milano, 1903. 8vo. 59 p.

NETTO (E.). See DÖLP (H.).

NÖLKE (F.). Uebersicht über die Theorie der Abelschen Funktionen zweier Variabeln. (Diss.) Marburg, 1903. 8vo. 38 pp.

ORLANDO (L.). Esercizi di calcolo infinitesimale. Messina, 1903. 16mo. 120 pp. Fr. 2.50

PASTORE (A.). Sopra la teoria della scienza: logica, matematica e fisica. Torino, 1903. 16mo. 31 + 238 pp. Fr. 4.00

PULITI (G.). See BALL (W. W. R.).

VERONESE (G.). Commemorazione del socio Luigi Cremona, letta dal socio Giuseppe Veronese nella seduta del 6 dicembre 1903. Roma, 1903. 4to. (*Rendiconti della r. Accademia dei Lincei*, Vol. XII, pp. 664–678.)

VOGT (H.). Eléments de mathématiques supérieures, à l'usage des physiciens, chimistes et ingénieurs et des élèves des facultés des sciences. 2e édition. Paris, Vuibert et Nony, 1904. 8vo. 7 + 619 pp.

WALLENBERG (G.). See JAHRBUCH.

II. ELEMENTARY MATHEMATICS.

ALLCOCK (C. H.). Theoretical geometry for beginners. Part 3. London and New York, Macmillan, 1904. 12mo. 113 pp. Cloth. $0.35

AMODEO (F.). Elementi di algebra. Parte I del volume II degli Elementi di matematica. Opera destinata alle scuole medie italiane. Napoli, Pierro, 1903. 16mo. 15 + 160 pp. Fr. 2.00

BAKER (W. M.) and BOURNE (A. A.). Key to elementary geometry. London, Bell, 1904. 8vo. Cloth. 6s.

BARNARD (S.) and CHILD (J. M.). New geometry for junior forms. London and New York, Macmillan, 1904. 12mo. 7 + 306 pp. Cloth. $0.60

BOULTON (E. S.). Geometry on modern lines for elementary students. London, Methuen, 1904. 8vo. 134 pp. Cloth. 2s.

BOURNE (A. A.). See BAKER (W. M.).

CAPELLI (A.). Elementi di aritmetica ragionata e di algebra ad uso dell' istruzione secondaria. Libro III: I numeri negativi. Napoli, Pellerano, 1904. 8vo. 8 + 112 pp.

CHILD (J. M.). See BARNARD (S.).

DALE (J. B.). Five-figure tables of mathematical functions: comprising tables of logarithms, powers of numbers, trigonometric, elliptic, and other transcendental functions. London, Arnold, 1904. 8vo. 108 pp. Cloth. 3s. 6d.

DEAKIN (R.). New matriculation algebra, tutorial algebra, elementary course. 4th impression, 3d edition, with a supplement on graphs. (University tutorial series.) London, Clive, 1904. 8vo. 512 pp. Cloth. 3s. 6d.

DEKKER (P.). See NIEMÖLLER (F.).

DOLEŽAL (E.). See STAMPFER (S.).

GAUTIER (D.). Division graphique des angles par les hyperboles étoilées. Paris, Hermann, 1904. 8vo. 44 pp.

GRÉVY (A.). Algèbre, à l'usage des élèves de la classe de quatrième B (programme du 31 mai 1902). 2e édition. Paris, Vuibert et Nony, 1904. 16mo. 337 pp.

——. Eléments de géométrie, à l'usage des élèves des classes de quatrième et troisième A (programmes du 31 mai 1902). 4e édition. Paris, Nony, 1903. 16mo. 172 pp.

——. Eléments de géométrie, à l'usage des élèves des classes de seconde et première A et B (programmes du 31 mai 1902). 4e édition. Paris, Nony, 1903. 16mo. 120 pp.

——. Géométrie, à l'usage des élèves des classes de cinquième B à troisième B (programmes du 31 mai 1902). 2e édition. Paris, Nony, 1903. 16mo. 306 pp. Fr. 2.25

——. Géométrie (compléments), à l'usage des élèves des classes de seconde et première C et D (programmes du 31 mai 1902). 2e édition. Paris, Nony, 1903. 16mo. 120 pp.

GUBLER (E.). Aufgaben aus der allgemeinen Arithmetik und Algebra für Mittelschulen. Heft 1: Resultate und Auflösungen. Zürich, Füssli, [1903]. 8vo. 29 pp. Boards. M. 1.50

LEIGHTON (L. R.). Supplementary and review work in algebra. Boston, Sibley, 1904. 16mo. 4 + 38 pp. $0.25

LUDLOW (H. H.). Geometric construction of the regular decagon and pentagon inscribed in a circle. Chicago, Open Court, 1904. 12mo. 12 pp.

M. (F. G.). Cours d'algèbre élémentaire, conforme aux derniers programmes de l'enseignement secondaire (1902). 3e édition. Paris, Poussielgue, 1904. 8vo. 8 + 568 pp.

MAJOCCHI (G.). Una linea retta punteggiata; costruzioni geometriche semplici delle equazioni di terzo grado colla linea retta e col circolo; moltiplicazione del cubo; trisezione dell'angolo. 2a edizione. Milano, Boccalari, 1903. 8vo. 24 pp., 1 plate. Fr. 1.00

MILNE (W. J.). Key to "Academic algebra" and "Advanced algebra." New York, American Book Company, 1904. 8vo. 486 pp. Cloth. $1.50

NIEMÖLLER (F.) und DEKKER (P.). Arithmetisches und algebraisches Unterrichtsbuch. Für den mathematischen Unterricht in der Mittel- und Oberstufe höherer Lehranstalten, unter Berücksichtigung der Bestimmungen der neuesten preussischen Lehrpläne bearbeitet. (In 4 Heften.) Heft 4: Pensum der beiden Primen des Realgymnasiums und der Oberrealschule. Breslau, Hirt, 1904. 8vo. 188 pp. Boards. M. 2.50

SCHMID (A.). See SICKENBERGER (A.).

SERRET (G. A.). Trattato di trigonometria. 6a edizione intieramente rifatta per cura di G. Tolomei. Firenze, Le Monnier, 1904. 16mo. 269 pp. Fr. 2.25

SHAW (W. N.). First lessons in observational geometry. London, Longmans, 1904. 8vo. 158 pp. Cloth. 2s.

SICKENBERGER (A.). Leitfaden der elementaren Mathematik. Teil 3: Stereometrie; Trigonometrie. 4te Auflage, bearbeitet von A. Schmid. München, Ackermann, 1904. 8vo. 5 + 104 pp. M. 1.20

STAMPFER (S.). Sechsstellige logarithmisch- trigonometrische Tafeln, nebst Hilfstafeln, einem Anhang und einer Anweisung zum Gebrauche der Tafeln. Neu bearbeitet von E. Dolezal. 20ste Auflage. Schulausgabe. Wien, Gerold, 1904. 8vo. 32 + 162 pp. Cloth. M. 3.00

TOLOMEI (G.). See SERRET (G. A.).

III. APPLIED MATHEMATICS.

ARNDT (L.). See GROSSMANN (J.).

BAUSCHINGER (J.). Ueber das Problem der Bahnverbesserung. Berlin, Dümmler, 1903. 4to. 35 pp., 1 plate. (Veröffentlichungen des k. astronomischen Rechen-Instituts zu Berlin, No. 23.) M. 2.00

——. See ENCKE (J. F.).

BLOCH (L.). Der Einfluss der Kurvenform auf die Wirkungsweise des Synchronmotors. (Diss.) Karlsruhe, 1903. 8vo. 76 pp. M. 2.40

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BRILLOUIN (M.). Propagation de l'électricité; histoire et théorie. Paris, Hermann, 1904. 8vo. 6 + 398 pp., 4 plates. (Cours du Collège de France.) Fr. 15.00

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CULMANN (P.), CZAPSKI (S.). See ROHR (M. VON).

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